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The influence of friction forces on the dynamics of a two-link mobile robot☆

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ABSTRACT

Controlled periodic motions of a planar two-link robot in a horizontal plane when there is dry friction are considered. The two-link is controlled by means of an internal torque applied to the joint connecting the links. The dynamics of the two-link, taking into account the influence of friction forces and the constrained nature of the control torque, is analysed assuming that the angle between the links is small. The conventional locomotion algorithm of a two-link is modified to ensure rectilinear displacement of the two-link. The influence of various geometrical and mechanical parameters of the system on the average rate of locomotion and on the power consumption during the motion of the two-link robot in a plane is investigated.

There is a great variety of modes of locomotion of bodies on a horizontal surface. Most of them are characterized by the fact that the points of contact of the body with the surface move constantly along the body. For example, the contact point moves continuously about a wheel, different links of the caterpillar track of a tractor contact the ground at different times, a walking apparatus is supported on the ground alternately by different legs, etc. However, snakes and certain creeping animals use another mode of locomotion, whose characteristic feature is contact between the surface and the body along practically its whole length during the entire time of motion. Several mechanical and biomechanical aspects of the motion of snakes have been examined.[1-4] Snakes move by bending their bodies in a horizontal plane. A planar multilink can serve as a mechanical analogue that is capable of locomoting in this manner. It has been shown[5-7] that a multilink system can move on a rough horizontal plane in any specified direction using only the internal control torques that are applied to the joints connecting two adjacent links, as well as the forces of dry friction between the multilink and the plane. The possible modes of locomotion of planar multilinks on a rough plane were investigated, their feasibility conditions were found, and the rates of locomotion were determined. The dependence of the average longitudinal speed of multilinks on the parameters of the system, namely, the length and mass of the links, the amplitude of the motions, and the friction coefficient, was investigated in Ref. 8. The parameters were optimized for a three-link and a two-link to achieve the maximum rate of longitudinal locomotion.

In the papers just cited the locomotion of the two-link and the three-link was built up from periodic motions with alternation of so-called slow and fast motions occurring during a period. During the slow motions of the two-link, it is assumed that one of the links (the body) remains fixed, while the other link (the tail) revolves about the central hinge. During the fast motions both links move, but the influence of the friction force may be neglected because the control torque is fairly large.

The experimental investigations in Ref. 9 demonstrated the practical feasibility of the proposed mode of locomotion. However, when the magnitude of the control torque is insufficiently large, the duration of the fast phases of motion is not very short, and the influence of the friction during the fast phases becomes significant.

In this paper, which continues the investigations in Refs 5-9, corrections to the motions of the two-link that appear because of the constrained nature of the control torque and the influence of the friction force during the fast phases of motion are determined. For simplicity, the angle between the robot links is assumed to be small. The influence of the parameters of the two-link (the lengths and masses of the links and the friction coefficient) on such important characteristics of the motion of the robot as the average rate of locomotion and the energy consumption per unit path length is analysed. This analysis enables an effective choice to be made of the parameters in the multicriteria problem.

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1. The Mechanical model of a two-link

Consider the two-link system $C_1C_0C_2$ shown in Fig. 1, which is oriented in a horizontal plane. We will call the link C_1C_0, which has a length a_1, the body, and we will call the link C_2C_0, which has a length a_2, the tail. For simplicity, we will assume that the links are rigid weightless rods and that all the mass is concentrated at the points C_0, C_1 and C_2. The masses of these points are equal to m_0, m_1 and m_2, respectively, and their friction coefficients with the surface are k_0, k_1 and k_2. The total mass of the two-link is $m = m_0 + m_1 + m_2$.

We introduce the Cartesian system of coordinates Oxy on the plane. The coordinates of the points of the two-link C_0, C_1 and C_2 will be denoted by x_0, y_0; x_1, y_1 and x_2, y_2, respectively. The angle between the body and the Ox axis will be denoted by θ, and the angle between the body and the tail will be denoted by α.

The coordinates of the body and the tail are written as follows:

$$\begin{aligned} &x_1 = x_0 + a_1\cos\theta, \qquad y_1 = y_0 + a_1\sin\theta \\ &x_2 = x_0 - a_2\cos(\theta+\alpha), \quad y_2 = y_0 - a_2\sin(\theta+\alpha) \end{aligned} \tag{1.1}$$

The coordinates of the centre of mass of the two-link x_c, y_c are expressed in the form

$$\begin{aligned} mx_c &= m_0x_0 + m_1x_1 + m_2x_2 = mx_0 + m_1a_1\cos\theta - m_2a_2\cos(\theta+\alpha) \\ my_c &= m_0y_0 + m_1y_1 + m_2y_2 = my_0 + m_1a_1\sin\theta - m_2a_2\sin(\theta+\alpha) \end{aligned} \tag{1.2}$$

The angular momentum of the robot about the point O, directed along a vertical axis perpendicular to the Oxy plane, is specified by the following expression

$$\begin{aligned} K &= m(x_0\dot{y}_0 - y_0\dot{x}_0) + m_1a_1(x_0\dot{\theta}\cos\theta + y_0\dot{\theta}\sin\theta) \\ &- m_2a_2\left[x_0(\dot{\theta}+\dot{\alpha})\cos(\theta+\alpha) + y_0(\dot{\theta}+\dot{\alpha})\sin(\theta+\alpha)\right] + m_1a_1(\dot{y}_0\cos\theta - \dot{x}_0\sin\theta) \\ &- m_2a_2\left[\dot{y}_0\cos(\theta+\alpha) - \dot{x}_0\sin(\theta+\alpha)\right] + m_1a_1^2\dot{\theta} + m_2a_2^2(\dot{\theta}+\dot{\alpha}) \end{aligned} \tag{1.3}$$

The control torque M acts at the hinge C_0. It can vary in an arbitrary manner, but its maximum value is limited to $M_{\max}$.

2. Elementary motions

The motions of the two-link are built up as a sequence of simple motions, which will be called elementary motions. All the elementary motions begin from a state of rest of the two-link and also end in a state of rest. The angle α between the body and the tail in each elementary motion varies monotonically in the range $(-\pi, \pi)$. We will denote the initial and final values of this angle in an elementary motion by α^0 and α^1, respectively. The elementary motions are subdivided into slow and fast motions. The time of a slow motion will be denoted by τ_s, and the time of a fast motion will be denoted by τ_f.

The slow motions are motions in which the body remains fixed and the tail turns through a certain angle. The conditions for immobility of the body during slow motions were obtained in Ref. 7. They have the form

$$\begin{aligned} &m_2a_2^2\varepsilon_0 + m_2gk_2a_2 \le m_1gk_1a_1 \\ &m_2a_2^2\varepsilon_0 + m_2gk_2a_2 + m_2a_1a_2[\omega_0^4 + (\varepsilon_0 + gk_2a_2^{-1})^2]^{1/2} \le m_0gk_0a_1 \end{aligned} \tag{2.1}$$

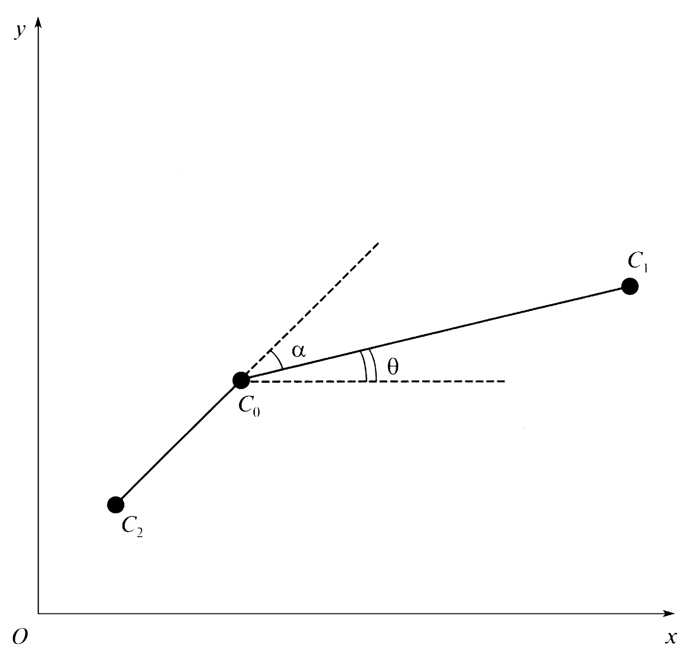

Fig. 1.

Here ω_0 and ε_0 are the maximum absolute values of the angular velocity and the angular acceleration of the tail during a slow motion. If a slow motion is performed with sufficiently small angular velocities and accelerations, these conditions take the simpler form

$$m_2k_2a_2 < m_1k_1a_1, \quad m_2k_2(a_1+a_2) < m_0k_0a_1 \tag{2.2}$$

We will use the term fast motions to refer to motions under which the magnitude of the control torque M is much greater than the torques created by the friction forces and the time of the motion τ_f is fairly short. For fast motions we have

$$|M| \gg m'gk'a', \quad m' = \max(m_1, m_2), \quad a'\max(a_1, a_2), \quad k' = \max(k_1, k_2), \quad \tau_f \ll \tau_s \tag{2.3}$$

We will use β to denote the maximum angle to which the tail is deflected, $\tau' = \sqrt{a'/(gk')}$ is the characteristic time for the system (the oscillation period of a pendulum that is similar in its parameters to the two-link), and ε' is the characteristic angular velocity of the tail during a fast motion. Then condition (2.3) can be rewritten in the form

$$M \sim J\varepsilon' \sim m'a'^2\varepsilon' \sim m'a'^2\beta/\tau_f^2 \gg m'a'gk' \tag{2.4}$$

where J is the characteristic moment of inertia. Hence we have $\beta \gg \tau_f^2/\tau'^2$. Since it is assumed in this paper that $\beta \ll 1$, the following quantitative estimate of the required shortness of the time τ_f holds

$$\tau_f \sim \beta\tau', \quad \tau' = \sqrt{a'/(gk')} \tag{2.5}$$

By virtue of conditions (2.3), when the fast motions are examined, the friction forces can be disregarded in a first approximation. Therefore, the laws of the conservation of momentum and angular momentum hold for these motions. However, at the start of a motion, the two-link is in a state of rest; therefore, for a fast motion we have

$$x_c \equiv \text{const}, \quad y_c \equiv \text{const}, \quad K \equiv 0$$

The expressions for x_c, y_c and K are given by relations (1.2) and (1.3). These conservation laws were used in Ref. [7]7 to determine the increments of the coordinates of the hinge and the angle θ in a fast motion when the angle α varies from α_0 to α_1. We have

$$\begin{aligned}
\Delta x_0 &= m^{-1}\{-m_1a_1[\cos(\theta+\Delta\theta)-\cos\theta] + m_2a_2[\cos(\theta+\Delta\theta+\alpha_1)-\cos(\theta+\alpha_0)]\} \\
\Delta y_0 &= m^{-1}\{-m_1a_1[\sin(\theta+\Delta\theta)-\sin\theta] + m_2a_2[\sin(\theta+\Delta\theta+\alpha_1)-\sin(\theta+\alpha_0)]\} \\
\Delta\theta &= -\int_{\alpha_0}^{\alpha_1}\varphi(\alpha)d\alpha = \gamma(\alpha_0)-\gamma(\alpha_1)
\end{aligned} \tag{2.6}$$

Here

$$\begin{aligned}
&\varphi(\alpha) = \frac{(m-m_2)m_2a_2^2 + D(\alpha)}{B_+ - C_+ + 2D(\alpha)}, \quad \gamma(\beta) = \int_0^\beta \varphi(\alpha)d\alpha = \frac{\beta}{2} + \frac{B_- + C_-}{A_+A_-}\operatorname{arctg}\left(\frac{A_+}{A_-}\operatorname{tg}\frac{\beta}{2}\right) \\
&A_\pm = [B_+ - (m_1a_1 \pm m_2a_2)^2]^{1/2}, \quad B_\pm = m(m_2a_2^2 \pm m_1a_1^2), \quad C_\pm = m_1^2a_1^2 \pm m_2^2a_2^2 \\
&D(\alpha) = m_1m_2a_1a_2\cos\alpha
\end{aligned} \tag{2.7}$$

Note that relations (2.6) and (2.7) only hold when the influence of the friction forces during the fast motions can be neglected. The influence of the friction forces introduces corrections to the increments of the coordinates obtained. We will estimate their order by taking the angle θ as an example. Taking into account relations (2.4) and (2.5), we have

$$\delta\theta \sim m'gk'a'\tau_f^2/J \sim (gk'/a')\tau_f^2 \sim \tau_f^2/\tau_0^2 \sim \beta^2$$

Similar estimates that are quadratic in β also hold for the corrections to the other variables.

3. Algorithm of the longitudinal locomotion of the two-link

We will describe the sequence of elementary motions that comprise the longitudinal locomotion of the two-link.

Suppose the two-link is at rest and has the form of a segment parallel to the x axis (state 0) in Fig. 2 at the initial time. We have $\theta = \alpha = 0$ in state 0. In addition, we take $x_0 = y_0 = 0$ in this state.

1. We perform a slow motion, during which the tail turns through an angle β and the body remains fixed. The two-link transfers into state 1 in Fig. 2, in which

$$\theta = 0, \quad \alpha = \beta, \quad x_0 = y_0 = 0$$

2. We perform a fast motion, during which the angle α varies from β to 0. The two-link transfers into state 2 in Fig. 2. In this state, according to relations (2.6), if the friction forces have no influence during the fast motion, we have

$$\begin{aligned}
&\theta = \gamma(\beta), \quad x_0 = m^{-1}[m_1a_1(1-\cos\gamma) + m_2a_2(\cos\gamma - \cos\beta)] \\
&y_0 = m^{-1}[-m_1a_1\sin\gamma + m_2a_2(\sin\gamma - \sin\beta)]
\end{aligned} \tag{3.1}$$

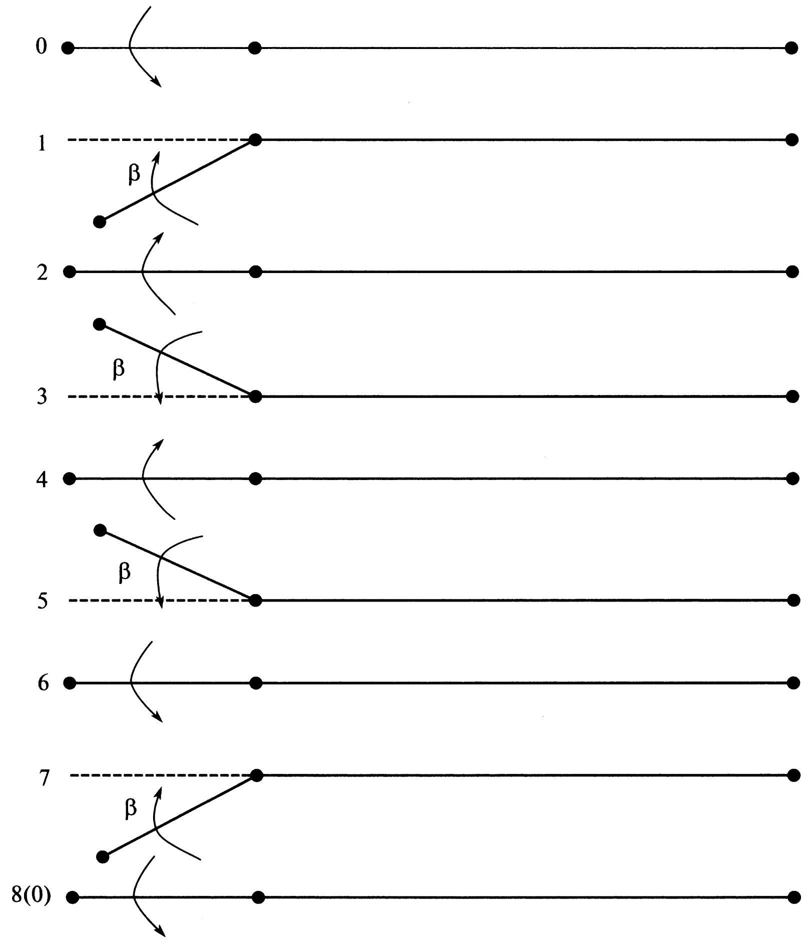

Fig. 2.

In Fig. 2 only the changes in the angle α are noted, and the changes in the angle θ and the coordinates x_0 and y_0 are not shown.

3. Using a slow motion, we vary the angle α from 0 to $-\beta$. The two-link transfers into state 3 in Fig. 2. The angle θ and the coordinates x_0 and y_0 remain unchanged and are specified by relations (3.1).
4. Using a fast motion, we vary the angle α from $-\beta$ to 0. The two-link transfers into state 4 in Fig. 2. In this state, according to relations (2.6), we have

$$\theta = 0, \quad x_0 = m^{-1}m_2a_2[\cos\gamma - \cos\beta + 1 - \cos(\gamma - \beta)]$$
$$y_0 = m^{-1}m_2a_2[\sin\gamma - \sin\beta - \sin(\gamma - \beta)]$$

Thus, the two-link is again a segment parallel to the Ox axis, but it moves forward and undergoes a lateral displacement ($y_0 \neq 0$). In order to eliminate this displacement, the motions performed are repeated in the reverse order, that is, motions 3, 4, 1 and 2 are performed (as shown in Fig. 2). After these motions are performed, the two-link compensates the lateral displacement, and the result of the entire sequence of motions is only locomotion in the longitudinal direction.

The displacement of the two-link along the Ox axis during the entire cycle amounts to[7]

$$l = 8m^{-1}m_2a_2\sin(\beta/2)\cos(\gamma(\beta)/2)\sin[(\beta - \gamma(\beta))/2] \tag{3.2}$$

The function $\gamma(\beta)$ is specified by relation (2.7).

The period of the motion equals

$$T = 4(\tau_s + \tau_f), \tau_s \gg \tau_f \tag{3.3}$$

The average speed is $\upsilon = 1/T$.

4. Slow and fast motions

We will specify the laws of variation of the angle α in the fast and slow phases of motion. We recall that the time of a slow motion is denoted by τ_s and the time of a fast motion is denoted by τ_f. We will assume that in both phases the angular acceleration is constant in

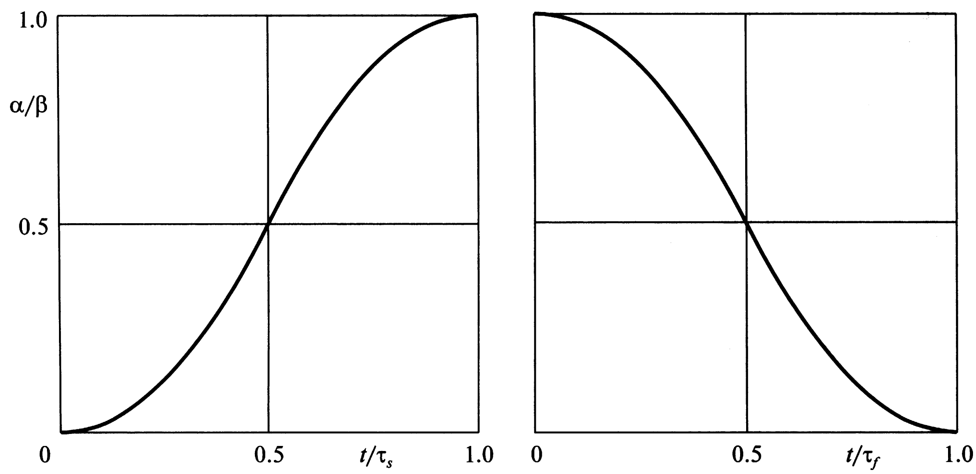

Fig. 3.

magnitude and changes sign in the middle of the phase, so that we have

$$\alpha(t) = \begin{cases} \varepsilon_s t^2/2, & 0 < t < \tau_s/2 \\ \varepsilon_s \tau_s^2/4 - \varepsilon_s (t - \tau_s)^2, & \tau_s/2 < t < \tau_s \end{cases} \tag{4.1}$$

$$\alpha(t) = \begin{cases} \varepsilon_f \tau_f^2/4 - \varepsilon_f t^2/2, & 0 < t < \tau_f/2 \\ \varepsilon_f (t - t_f)^2/2, & \tau_f/2 < t < \tau_f \end{cases} \tag{4.2}$$

for slow and fast motions, respectively. Here ε_s and ε_f are the maximum absolute values of the angular acceleration for the slow and fast phases, respectively. For the maximum absolute values of the angular velocity during these motions, we have $\omega_s = \varepsilon_s \tau_s/2$ and $\omega_f = \varepsilon_f \tau_f/2$.

Since the total change in the angle $\alpha(t)$ during each elementary motion is equal to β, the following relations hold:

$$\beta = \varepsilon_s \tau_s^2/4 = \varepsilon_f \tau_f^2/4 \tag{4.3}$$

Fig. 3 shows the law of variation of the angle α (relation (4.1) in the slow phase on the left and relation (4.2) in the fast phase on the right).

5. Lagrange's equations and their linearization

In the generalized coordinates

$$q_1 = x_0, \quad q_2 = y_0, \quad q_3 = \theta, \quad q_4 = \alpha$$

we write Lagrange's equations for the two-link

$$\begin{aligned} & m\ddot{x}_0 - m_1 a_1 \ddot{\theta} \sin\theta - m_1 a_1 \dot{\theta}^2 \cos\theta + m_2 a_2 (\ddot{\theta} + \ddot{\alpha}) \sin(\theta + \alpha) \\ & + m_2 a_2 (\dot{\theta} + \dot{\alpha})^2 \cos(\theta + \alpha) = Q_x \\ & m\ddot{y}_0 + m_1 a_1 \ddot{\theta} \cos\theta - m_1 a_1 \dot{\theta}^2 \sin\theta - m_2 a_2 (\ddot{\theta} + \ddot{\alpha}) \cos(\theta + \alpha) \\ & + m_2 a_2 (\dot{\theta} + \dot{\alpha})^2 \sin(\theta + \alpha) = Q_y \\ & m_1 a_1^2 \ddot{\theta} + m_2 a_2^2 (\ddot{\theta} + \ddot{\alpha}) + m_1 a_1 \ddot{y}_0 \cos\theta - m_1 a_1 \ddot{x}_0 \sin\theta \\ & + m_2 a_2 \ddot{x}_0 \sin(\theta + \alpha) - m_2 a_2 \ddot{y}_0 \cos(\theta + \alpha) = Q_\theta \\ & m_2 a_2^2 (\ddot{\theta} + \ddot{\alpha}) + m_2 a_2 [\ddot{x}_0 \sin(\theta + \alpha) - \ddot{y}_0 \cos(\theta + \alpha)] = Q_\alpha \end{aligned} \tag{5.1}$$

Here Q_x, Q_y, Q_θ and Q_α are the generalized forces specified by the dry friction forces and the control moment in the hinge.

To simplify the analysis, we will replace the third equation in the system by the difference between the third and fourth equations:

$$m_1 a_1^2 \ddot{\theta} + m_1 a_1 (\ddot{y}_0 \cos\theta - \ddot{x}_0 \sin\theta) = Q_\theta - Q_\alpha$$

We denote the projections of the friction forces acting at the point C_i of the two-link onto the coordinate axes as $F_x{}^i$ and $F_y{}^i$, respectively (i = 0, 1, 2). Then the expressions for the generalized forces take the form

$$
\begin{aligned}
&Q_x = F_x^0 + F_x^1 + F_x^2, \quad Q_y = F_y^0 + F_y^1 + F_y^2 \\
&Q_\theta = -F_x^1 a_1 \sin\theta + F_y^1 a_1 \cos\theta + F_x^2 a_2 \sin(\theta+\alpha) - F_y^2 a_2 \cos(\theta+\alpha) \\
&Q_\alpha = F_x^2 a_2 \sin(\theta+\alpha) - F_y^2 a_2 \cos(\theta+\alpha) + M
\end{aligned}
\tag{5.2}
$$

Here, by Coulomb's law, we have

$$F_x^i = -\frac{gk_i m_i \dot{x}_i}{\sqrt{\dot{x}_i^2 + \dot{y}_i^2}}, \quad F_y^i = -\frac{gk_i m_i \dot{y}_i}{\sqrt{\dot{x}_i^2 + \dot{y}_i^2}}, \quad i = 0, 1, 2$$

We will now determine the order of magnitude of the friction forces that appear when there is a small change in the angle α. For this purpose, we will estimate how great the increments of the coordinates x_0 and y_0 are from conditions (2.6). Substituting the values

$$\alpha_0 = 0, \quad \theta = 0, \quad \alpha_1 = \beta, \quad \Delta\theta = -\gamma(\beta)$$

we obtain the following result

$$\Delta x_i = O(\beta^2), \quad \Delta y_i = O(\beta)$$

Thus, in order of magnitude we have $\Delta x_i \sim \beta \Delta y_i$. A similar relation holds for the velocities, and thus $\dot{x}_i \sim \beta \dot{y}_i$. Therefore, for the friction forces we have

$$F_x^i = O(\beta), \quad F_y^i = O(1)$$

Now we change to dimensionless variables using the replacements

$$
\begin{aligned}
&\alpha = \beta\alpha', \quad \theta = \beta\theta', \quad x_0 = \beta a_1 x\alpha_0', \quad y_0 = \beta a_1 y_0', \quad t = \tau_0 t', \quad \tau_0 = \sqrt{\frac{a_1}{gk_0}} \\
&F_x^i = m_0 g F_x'^i, \quad F_y^i = m_0 g F_y'^i, \quad M = m_0 g a_1 M' \\
&\mu = \frac{m}{m_0}, \quad \mu_1 = \frac{m_1}{m_0}, \quad \mu_2 = \frac{m_2}{m_0}, \quad \lambda = \frac{a_2}{a_1}
\end{aligned}
\tag{5.3}
$$

During the fast and slow motions, α varies in the range from 0 to 1.

After making these replacements, we linearize Lagrange's equations, assuming that the angle β is small, and we omit all the primes on the variables introduced. Note further that $\beta F_x{}^i = O(\beta^2)$. After simplifying, we obtain

$$
\begin{aligned}
&k_0\beta\mu\ddot{x}_0 = F_x^0 + F_x^1 + F_x^2, \quad k_0\beta\left[\mu\ddot{y}_0 - \ddot{\alpha}\lambda\mu_2 + \ddot{\theta}(\mu_1 - \lambda\mu_2)\right] = F_y^0 + F_y^1 + F_y^2 \\
&k_0\beta\mu_1\left(\ddot{\theta} + \ddot{\alpha}\right) = F_y^1 - M, \quad k_0\beta\mu_2\lambda[\lambda(\ddot{\theta} + \ddot{\alpha}) - \ddot{y}_0] = M - \lambda F_y^2
\end{aligned}
\tag{5.4}
$$

We will examine these equations for the slow and fast motions, confining ourselves to an analysis of only the lateral and angular motions. Thus, the longitudinal motions will henceforth not be considered, and the last three equations in (5.4) will be analysed.

6. Influence of the friction force on the locomotion of the two-link during slow motion

We will consider the slow motion of the two-link robot from the state $y_0 = 0, \theta = 0, \alpha = 0$ to the state $y_0 = 0, \theta = 0, \alpha = \beta$ and estimate the time needed to complete this motion. We will consider the motion in the linear approximation with respect to β in the dimensionless variables (5.3).

In the slow phase $x_0(t) = y_0(t) = \theta(t) = 0$; therefore, the linearized Lagrange equations (5.4) reduce to the single equation:

$$k_0\beta\mu_2\lambda^2\ddot{\alpha} = M - F_y^2\lambda \quad F_y^2 = -k_2\frac{\dot{y}_2}{\sqrt{\dot{x}_2^2 + \dot{y}_2^2}} \approx k_2\cos\alpha \approx k_2$$

Hence we have

$$k_0\beta\mu_2\lambda^2\ddot{\alpha} = M - k_2 \tag{6.1}$$

In the dimensionless variables (5.3) the law of variation (4.1) of the angle α during the time of a slow motion τ_s is written in the form

$$\alpha(t) = \begin{cases} \varepsilon_s t^2/(2\tau_s^2), & 0 < t < \tau_s/2 \\ \varepsilon_s/4 - \varepsilon_s(t-\tau_s)^2/(2\tau_s^2), & \tau_s/2 < t < \tau_s \end{cases} \tag{6.2}$$

Here τ_s and ε_s are also dimensionless, and $\varepsilon_s = 4$, since the angle α should vary from 0 to 1.

For the law of variation (6.2) of the angle α we have

$$\max_t \ddot{\alpha} = \varepsilon_s/(2\tau_s^2) = 2/\tau_s^2, \quad \max_t \dot{\alpha} = 2/\tau_s, \quad t \in (0, \tau_s) \tag{6.3}$$

and from Eq. (6.1) we obtain

$$M(t) = k_2 - 4\,\mathrm{sign}\left(t - \frac{\tau_s}{2}\right) k_0\beta\mu_2\lambda^2/\tau_s^2, \quad M_{\max}^s = k_2 + 4k_0\beta\mu_2\lambda^2/\tau_s^2$$

Substituting expressions (6.3) into dimensionless inequalities (2.1), we find the conditions that specify the minimum time of motion in the slow phase. We obtain the shortest possible time of slow motion

$$\tau_s^{\min} = \sqrt{4k_0\beta\mu_2\lambda}\max\left(\sqrt{\frac{\lambda}{k_1\mu_1 - k_2\mu_2\lambda}}, \sqrt{\frac{\lambda+1}{k_0 - k_2\mu_2(\lambda+1)}}\right) \tag{6.4}$$

and it corresponds to the maximum attainable dimensionless torque for slow motion

$$M_{\max}^s = k_2 + 4k_0\beta\mu_2\lambda^2/\tau_s^2$$

The radicands in equality (6.4) must be positive. This is equivalent to conditions (2.2).

7. The Influence of the friction force on the locomotion of the two-link during fast motion

We will consider the fast motion of the two-link robot from the state $y_0 = 0$, $\theta = 0$, $\alpha = \beta$ to the state $y_0 = \Delta y_0$, $\theta = \Delta\theta$, $\alpha = 0$ and estimate the corrections to the final position $\delta\theta$ and δy_i, which appear due to the finite time of a fast motion τ_f. They are caused by the constrained nature of the maximum torque created by the actuator located on the central link: $M_{\max} = \max M(t)$ at $t \in [0, \tau_f]$. As before, we will use an approximation that is linear in the small angle β and perform the calculations in the dimensionless variables defined by equalities (5.3).

We will first solve the problem when there are no friction forces. After solving the problem for the accelerations, the last three equalities in (5.4) take the form

$$\beta\ddot{y}_0 = \frac{\lambda+1}{k_0\lambda}M = G_y M, \quad \beta\ddot{\theta} = -\frac{\mu_1\lambda + \lambda + \mu_1}{\lambda k_0\mu_1}M = G_\theta M$$
$$\beta\ddot{\alpha} = \frac{\mu_2\lambda^2 + \mu_1(\mu_2(\lambda+1)^2 + 1)}{\lambda^2 k_0\mu_1\mu_2}M = G_\alpha M \tag{7.1}$$

As in the case of slow motion, we assign the law of variation of the angle α during the time of fast motion τ_f in the form

$$\alpha(t) = \begin{cases} \varepsilon_f/4 - \varepsilon_f t^2/(2\tau_f^2), & 0 < t < \tau_f/2 \\ \varepsilon_f(t-\tau_f)^2/(2\tau_f^2), & \tau_f/2 < t < \tau_f \end{cases}$$

Then, from the last equation in (7.1) we obtain the law of variation of the control torque

$$M = \frac{4\beta}{G_\alpha\tau_f^2}\mathrm{sign}\left(t - \frac{\tau_f}{2}\right)$$

Thus, $M^0_{\max} = 4\beta/(G_\alpha\tau_f^2)$. We find the dependence of the time of fast motion on the maximum torque:

$$\tau_f^2 = \frac{4k_0\mu_1\mu_2\lambda^2}{\mu_2\lambda^2 + \mu_1(\mu_2(\lambda+1)^2 + 1)}\frac{\beta}{M_{\max}^0} \tag{7.2}$$

Knowing M, from Eqs (7.1) we can easily find the increments of the coordinates of the two-link after the time of fast motion τ_f when there are no friction forces:

$$\Delta_0 y_0 = \frac{G_{y_0}}{G_\alpha}, \quad \Delta_0\theta = \frac{G_\theta}{G_\alpha} \tag{7.3}$$

It can be verified that the linearization of expressions (2.6) leads to the same increments, which confirms the correctness of the linearization of the original system.

Now we introduce friction forces into the system in an approximation that is linear in β:

$$F_y^0 = -k_0, \quad F_y^1 = -k_1, \quad F_y^2 = k_2 \tag{7.4}$$

The signs were set in accordance with the fact that during the motion under consideration $\dot{y}_0 < 0$, $\dot{y}_1 < 0$, and $\dot{y}_2 < 0$.

The last three equalities in (5.4) take the form

$$k_0\beta\left[\mu\ddot{y}_0 - \ddot{\alpha}\lambda\mu_2 + \ddot{\theta}(\mu_1 - \lambda\mu_2)\right] = k_2 - k_1 - k_0$$
$$k_0\beta\mu_1\left(\ddot{\theta} + \ddot{\alpha}\right) = -M - k_1, \quad k_0\beta\mu_2\lambda[\lambda(\ddot{\theta} + \ddot{\alpha}) - \ddot{y}_0] = M - k_2$$

We will solve these equations for the second derivatives:

$$\beta\ddot{y}_0 = r_{y_0} + G_y M, \quad \beta\ddot{\theta} = r_\theta + G_\theta M, \quad \beta\ddot{\alpha} = r_\alpha + G_\alpha M \tag{7.5}$$

For the coefficients we have

$$r_{y_0} = -1, \quad r_\theta = \frac{k_1 + k_0\mu_1}{k_0\mu_1}, \quad r_\alpha = -\frac{k_2\mu_1 + \left(k_1\lambda + k_0\mu_1(\lambda+1)\right)\mu_2}{k_0\mu_1\mu_2\lambda} \tag{7.6}$$

Adopting the prior law of variation of the angle $\alpha(t)$, for the control torque we obtain the expression

$$M = r_\alpha + \frac{4\beta}{G_\alpha\tau_f^2}\operatorname{sign}\left(t - \frac{\tau_f}{2}\right)$$

For the maximum absolute value of the torque, we have

$$M_{\max} = r_\alpha + \frac{4\beta}{G_\alpha\tau_f^2}$$

The increments of the coordinates of the two-link are specified, in turn, by the expressions

$$\Delta y_0 = \frac{G_{y_0}}{G_\alpha} + (r_{y_0} + G_{y_0}r_\alpha)\frac{\tau_f^2}{2\beta}, \quad \Delta\theta = \frac{G_\theta}{G_\alpha} + (r_\theta + G_\theta r_\alpha)\frac{\tau_f^2}{2\beta} \tag{7.7}$$

Therefore, the corrections to the changes in the coordinates that appear due to the friction forces are written as follows:

$$\delta y_0 = \Delta y_0 - \Delta_0 y_0 = (r_{y_0} + G_{y_0}r_\alpha)\frac{\tau_f^2}{2\beta}, \quad \delta\theta = \Delta\theta - \Delta_0\theta = (r_\theta + G_\theta r_\alpha)\frac{\tau_f^2}{2\beta} \tag{7.8}$$

If we substitute the time of fast motion τ_f as a function of the maximum torque $M_{\max}$ into these expressions, we obtain

$$\delta y_0 = 2\frac{r_{y_0} + G_{y_0}r_\alpha}{G_\alpha(M_{\max} - r_\alpha)}, \quad \delta\theta = 2\frac{r_\theta + G_\theta r_\alpha}{G_\alpha(M_{\max} - r_\alpha)} \tag{7.9}$$

Thus, in dimensional coordinates the corrections to the increments of the lateral displacement and the angle θ are of the order of $O(\beta/M_{\max})$. Since, according to relations (2.3), the value of $M_{\max}$ is large (for example, of the order of $1/\beta$), these corrections become quadratic in β. In this case the influence of the friction force can be neglected. If $M_{\max} = O(1)$ (but is large enough to overcome the static friction force for all the links in the two-link), the influence of the friction force becomes linear in β.

After a second sequence of slow and fast phases, during which the angle α varies from $-\beta$ to 0, the corrections found simply change sign, because the expressions for the friction forces remain unchanged (only the sign changes) as a result of the linear additions to the zero values of y_0 and θ that appear after the first sequence.

Thus, even the unmodified locomotion algorithm of the two-link[7] leads to elimination of the corrections to the changes in its coordinates in the linear approximation.

8. Energy consumption per unit of path length and average speed of the two-link

We recall that the dimensionless parameters of the system specified by the last three formulae in (5.3) are used. We introduce the dimensionless energy as $E = m_0 g a_1 \beta E'$ (the prime will henceforth be omitted). We will also assume that the friction coefficients k_0, k_1 and k_2 lie in the interval $[k_-, k_+]$. All the calculations will be performed on the assumption that the maximum angle of deflection of the tail of the two-link β is small.

As the two-link moves, energy is expended during the slow and fast motions to overcome the friction forces. We will find the energy consumption during one cycle of slow and fast motions. During one slow motion, the energy expended by the tail is

$$\Delta E_s = F_y^2 \Delta y_2^s = k_2\lambda$$

During one fast motion, the energy expended by the tail (we use formulae (7.3) for the coordinate increments) is

$$\Delta E_f = F_y^0\Delta y_0^f + F_y^1\Delta y_1^f + F_y^1\Delta y_1^f = \lambda\frac{k_1\mu_2\lambda + \mu_1\left(k_2 + k_0\mu_2(\lambda+1)\right)}{\mu_2\lambda^2 + \mu_1 + \mu_1\mu_2(\lambda+1)^2}$$

Motion along the Ox axis is ignored when calculating the energy consumption because the losses in this motion are an order of magnitude lower than the losses in the motion of the parts of the two-link along the Oy axis. The corrections (7.9) to the displacements of the parts of the two-link that appear due to the constrained nature of the control torque will also be neglected in view of their small magnitude.

After a complete cycle of motions by the tail of the two-link described in Section 4, the displacement of the robot in the direction of motion is (see dimensionless expression (3.2), which has been linearized in β)

$$l = \frac{2\beta\mu_1\mu_2\lambda\left(\mu_2(\lambda+1)+1\right)}{\mu\left(\mu_2\lambda^2 + \mu_1 + \mu_1\mu_2(\lambda+1)^2\right)}$$

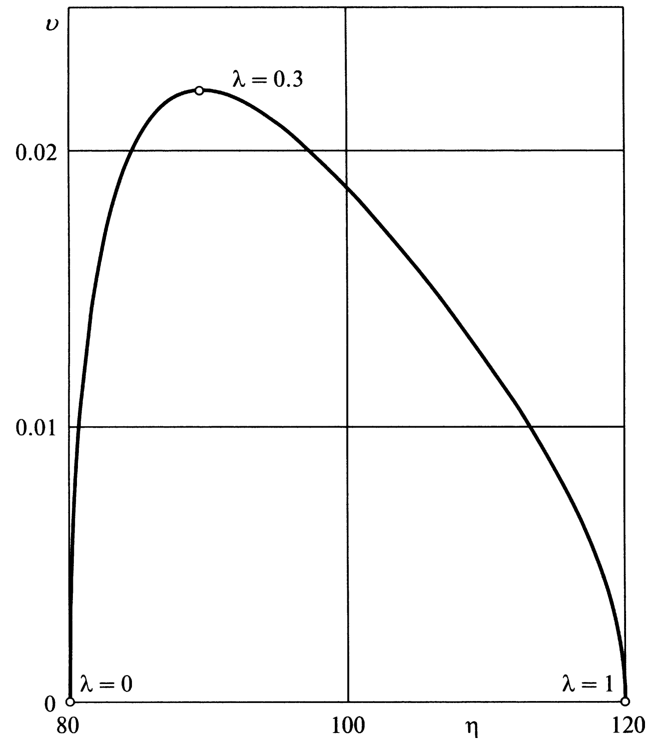

Fig. 4.

The energy consumption per unit of path length during the complete cycle of motions equals

$$\eta = 4\frac{\Delta E_s + \Delta E_f}{l} = 2\mu\frac{k_1\lambda + k_0\mu_1(\lambda+1) - k_2\left(\lambda^2 + \mu_1(\lambda+1)^2\right)}{\beta\mu_1(\mu_2\lambda + \mu_2 + 1)} \tag{8.1}$$

As expected, the energy consumption increases as the angle β decreases. We note at once that the expression is linear in the friction coefficients. For this reason, for the smallest energy consumption, the friction coefficients should have the smallest attainable values, but they should be such that conditions (2.2), which ensure the possibility of slow motions, are satisfied.

The average speed of the two-link is specified by the expression

$$\upsilon = l/(\tau_s + \tau_f)$$

The values of τ_s and τ_f are specified by expression (6.4) and (7.2), respectively. Since we decided to neglect the influence of the constrained nature of the control torque on the fast motion, the relation $\tau_f \sim \beta\tau_s$ holds. Therefore, in a first approximation we have

$$v = l\left[\sqrt{4k_0\beta\mu_2\lambda}\min\left(\sqrt{\frac{\lambda}{k_1\mu_1 - k_2\mu_2\lambda}}, \sqrt{\frac{\lambda+1}{k_0 - k_2\mu_2(\lambda+1)}}\right)\right]^{-1} \tag{8.2}$$

As the angle β is increased, the speed increases as $\sqrt{\beta}$. Therefore, to increase the average speed and to reduce the energy consumption, the angle β must be increased. It was previously[8] found that the maximum speed of a two-link in the case of an unconstrained control moment is obtained when $k_2 = k_-$ and $k_0 = k_1 = k_+$ for an arbitrary (not small) value of β. Therefore, a decrease in k_2 also causes an increase in speed and a reduction in energy consumption.

We will investigate numerically the dependences of the average speed and the energy consumption on the following parameters of the system: the length ratio, the mass ratio and the friction coefficients.

We choose the following starting values for the parameters

$$\beta = 0.1, \quad \lambda = 0.5, \quad k_0 = k_1 = k_2 = 0.4, \quad \mu_2 = 0.5, \quad \mu_1 = 1 \tag{8.3}$$

Then, we vary one of the parameters in burn, while fixing the starting values of the remaining parameters in (8.3), and construct a graph by plotting the energy consumption along the horizontal axis and the average speed along the vertical axis.

In Fig. 4 the parameter λ, which is the ratio of the lengths of the tail and the body, varies in the range from 0 to 1. It can be seen that there is a value $\lambda \approx 0.3$, above which the energy consumption increases and the speed drops, i.e., it would be pointless to increase λ above this value.

In Fig. 5, the parameter μ_1, which is the ratio between the masses of the body and the central hinge, varies in the range from 0.25 to 5, and μ_2, which is the ratio between the masses of the tail and the central joint, varies in the range from 0.1 to 2/3. The starting value $\mu_2 = 0.5$ was fixed on the graph for μ_1. Conversely, the value $\mu_1 = 1$ was fixed on the graph for μ_2. The best value $\mu_1 \approx 0.25$ is very close to the

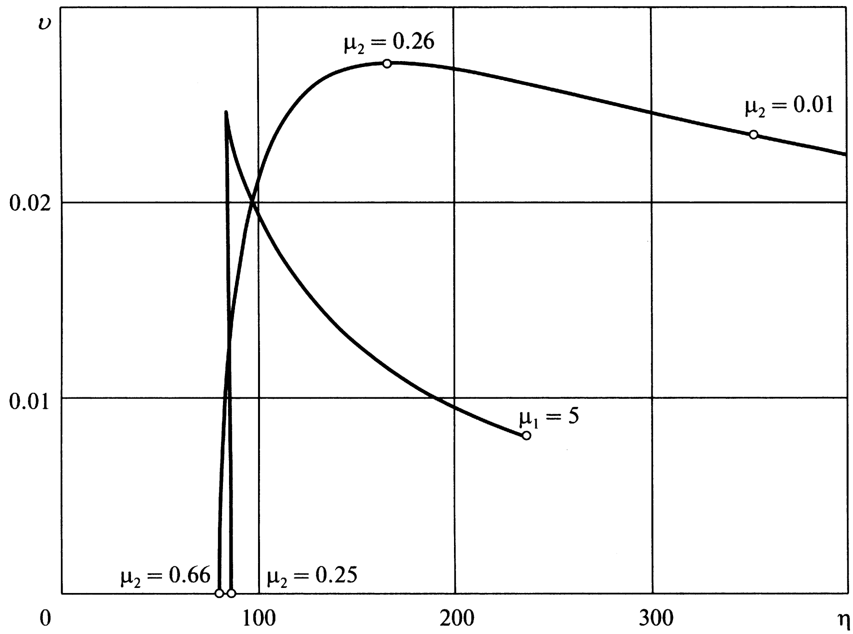

Fig. 5.

critical value, after which completion of the slow motions is impossible. There is a value $\mu_2 \approx 0.26$, above which the energy consumption increases and the speed drops as μ_2 is increases.

In Fig. 6 the parameter k_0, which is the friction coefficient of the central hinge, varies in the range from 0.3 to 0.9, and k_1, which is the friction coefficient of the body, varies in the range from 0.1 to 0.9. The value $k_0 = 0.3$ is the minimum admissible value for this friction coefficient. The value $k_1 = 0.1$ is the minimum admissible value for this friction coefficient. After the value $k_1 \approx 0.14$, as this friction coefficient is increased, the speed increases only slightly, while the energy consumption increases rapidly.

The kinks on the graphs are associated with the transition in expression (6.4) for the time of a slow motion from one argument of the maximum to another.

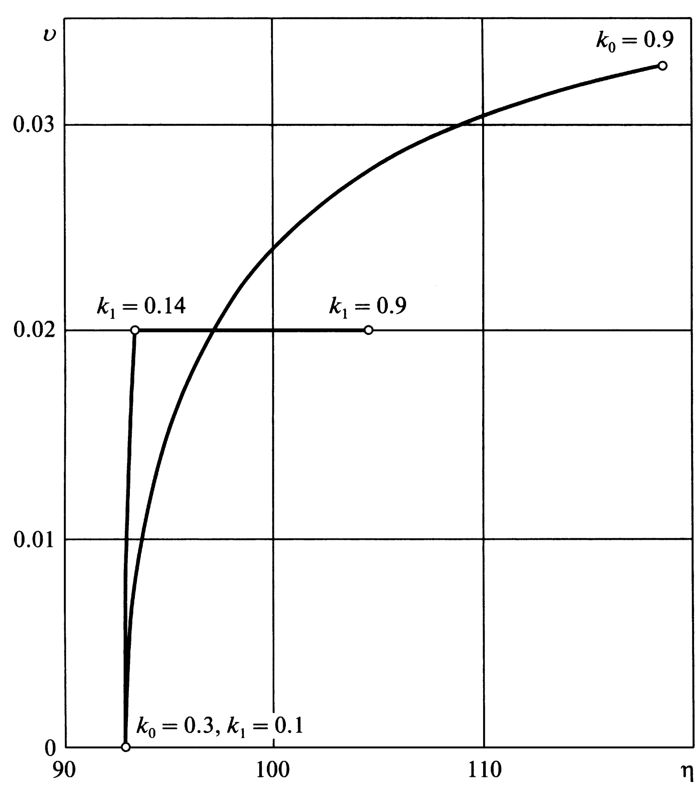

Fig. 6.

9. Conclusions

The influence of the friction forces on the dynamics of the snake-like motions of a two-link in a plane with a simple control algorithm based on alternating fast and slow phases of motion has been analysed. It has been shown, in the case of a small deflection angle of the tail of the two-link, that the corrections to the lateral displacement and the angle of rotation of the body are small high-order corrections. The maximum possible torque that can be applied to the central joint without violating the feasibility conditions of the slow motions has been determined. The influence of the friction forces has been taken into account in determining the maximum value of the torque required to perform a fast motion in a specified time. The average speed and energy consumption during motion of the two-link have been estimated as functions of various characteristic parameters, including the friction coefficients, geometrical parameters and mass parameters. Guidelines regarding the ineffectiveness of using specific ranges of values for designing efficient two-link robots have been presented for certain parameters.

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